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Non-dimensional governing equations are

$$u_t = \frac{f}{\omega} v - p_x - \frac{\alpha}{\omega} u, \tag{1}$$

$$v_t = -\frac{f}{\omega} u - \frac{\alpha}{\omega} v, \tag{2}$$

$$\frac{\omega^2}{N^2} w_t = b - p_z - \frac{\alpha}{\omega}\frac{\omega^2}{N^2} w, \tag{3}$$

$$b_t + w = Q - \frac{\alpha}{\omega} b, \tag{4}$$

$$u_x + w_z = 0, \tag{5}$$

$$w(z=\sigma(x)) = M u(z=\sigma(x)), \tag{6}$$

where $\sigma(x) = Mx$ is the topography height with $M \in \mathbb{R}$ constant, choosing coordinates so $M \geq 0$. Note $\omega \in \mathbb{R}$, $\omega > 0$ is the frequency of the forcing in dimensional coordinates, after non-dimensionalization we have $Q = H(x,z)\cos(t) = \frac{1}{2}H(x,z)\left[e^{it} + e^{-it}\right]$.

Define ψ such that $(u,w) = (\psi_z, -\psi_x)$.

Note (1) and (2) imply

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)u = \frac{f}{\omega} v - p_x, \tag{7}$$

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)v = -\frac{f}{\omega} u. \tag{8}$$

Taking $\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)$ of (7) and subbing in

(8) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u = -\frac{f^2}{\omega^2}u - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_x \quad (9)$$

Cross-differentiating (3) and (9) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_{xz}, \quad (10)$$

$$\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x = b_x - p_{zx} \quad (11).$$

Using (11) to substitute p_{xz} in (10) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)\left(\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x - b_x\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x \quad (12).$$

Note (4) implies

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b + w = Q$$

$$\Rightarrow -\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x = w_x - Q_x. \quad (13)$$

Subbing (13) into (12) and then subbing ψ gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x + w_x - Q_x$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{zz} + \frac{f^2}{\omega^2}\psi_{zz} + \frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{xx} + \psi_{xx} = -Q_x$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2+\frac{f^2}{\omega^2}\right]\psi_{zz} + \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2+1\right]\psi_{xx} = -Q_x \quad (14).$$

At this point we would typically take a Fourier transform in x, but we cannot do this here due to the sloped lower boundary. Thus we define new coordinates

$x_\sigma := x$ and $z_\sigma := z - \sigma(x) = z - Mx$.

The multivariate chain rule then implies

$$\frac{\partial}{\partial x} = \frac{\partial x_\sigma}{\partial x}\frac{\partial}{\partial x_\sigma} + \frac{\partial z_\sigma}{\partial x}\frac{\partial}{\partial z_\sigma} = \frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}, \quad (15)$$

$$\frac{\partial}{\partial z} = \frac{\partial x_\sigma}{\partial z}\frac{\partial}{\partial x_\sigma} + \frac{\partial z_\sigma}{\partial z}\frac{\partial}{\partial z_\sigma} = \frac{\partial}{\partial z_\sigma}. \quad (16)$$

Subbing (15) and (16) into (14) gives

$$\left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + \frac{\delta^2}{\omega^2}\right]\psi_{z_\sigma z_\sigma} + \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]\left[\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right]^2\psi$$
$$= -\left(\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right)Q$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + \frac{\delta^2}{\omega^2}\right]\psi_{z_\sigma z_\sigma}$$
$$+ \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]\left(\frac{\partial^2}{\partial x_\sigma^2} - 2M\frac{\partial^2}{\partial x_\sigma \partial z_\sigma} + M^2\frac{\partial^2}{\partial z_\sigma^2}\right)\psi$$
$$= -\left(\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right)Q$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\left(1 + M^2\frac{\omega^2}{N^2}\right) + \frac{\delta^2}{\omega^2} + M^2\right]\psi_{z_\sigma z_\sigma}$$
$$- \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]2M\,\psi_{x_\sigma z_\sigma}$$
$$+ \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]\psi_{x_\sigma x_\sigma}$$
$$= -\left(\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right)Q \quad (17)$$

Recalling that $Q = H(x,z)\,\frac{e^{it}+e^{-it}}{2}$, we can solve for the responses to $\frac{1}{2}He^{it}$ and $\frac{1}{2}He^{-it}$ seperately, then sum these responses. Consider first the $\frac{1}{2}He^{it}$ mode. Then we must have

$\psi = \tilde{\psi}(x,z)e^{it}$. Subbing $\tilde{\psi}(x,z)e^{it}$ into (17) and simplifying we have

$$\left[\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)\left(1+M^2\frac{\omega^2}{N^2}\right)+\frac{\delta^2}{\omega^2}+M^2\right]\psi_{z_0z_0}$$
$$-\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]2M\psi_{x_0z_0}$$
$$+\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]\psi_{x_0x_0}$$
$$=-\left(\frac{\partial}{\partial x_0}-M\frac{\partial}{\partial z_0}\right)\frac{H}{2}. \qquad (18)$$

Taking the Fourier transform in x_0 of both sides of (18) then gives

$$\left[\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)\left(1+M^2\frac{\omega^2}{N^2}\right)+\frac{\delta^2}{\omega^2}+M^2\right]\hat{\psi}_{z_0z_0}$$
$$-ih\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]2M\hat{\psi}_{z_0}$$
$$-h^2\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]\hat{\psi}$$
$$=-\left(ih-M\frac{\partial}{\partial z_0}\right)\frac{\hat{H}}{2}$$

$$\Rightarrow \left[\left(1-2i\frac{\alpha}{\omega}-\frac{\alpha^2}{\omega^2}\right)\left(1+M^2\frac{\omega^2}{N^2}\right)-\frac{\delta^2}{\omega^2}-M^2\right]\hat{\psi}_{z_0z_0}$$
$$+ih\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]2M\hat{\psi}_{z_0}$$
$$+h^2\left[\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)+1\right]\hat{\psi}$$
$$=\frac{1}{2}\left(ih-M\frac{\partial}{\partial z_0}\right)\hat{H}. \qquad (19)$$

To simplify notation, let

$$B := \left[\left(1-2i\frac{\alpha}{\omega}-\frac{\alpha^2}{\omega^2}\right)\left(1+M^2\frac{\omega^2}{N^2}\right)-\frac{\delta^2}{\omega^2}-M^2\right]^{\frac{1}{2}},$$

$$C := \left[1+\frac{\omega^2}{N^2}\left(-1+2i\frac{\alpha}{\omega}+\frac{\alpha^2}{\omega^2}\right)\right]^{\frac{1}{2}},$$

$A := B/C$. Subbing A, B, C into (19) gives

$$B^2 \hat{\varphi}_{z_0 z_0} + ik\,2M C^2 \hat{\varphi}_{z_0} + k^2 C^2 \hat{\varphi} = \tfrac{1}{2}\left(ik - M\tfrac{\partial}{\partial z_0}\right)\hat{H}$$

$$\Rightarrow \hat{\varphi}_{z_0 z_0} + ik\frac{2M}{A^2}\hat{\varphi}_{z_0} + \frac{k^2}{A^2}\hat{\varphi} = \frac{1}{2B^2}\left(ik - M\frac{\partial}{\partial z_0}\right)\hat{H} \quad (20).$$

To solve (20) we use Green's method. First consider the corresponding homogeneous equation, which has solutions proportionate to e^{imz_0}. Thus

$$-m^2 + ik\frac{2M}{A^2}im + \frac{k^2}{A^2} = 0$$

$$\Rightarrow m^2 + km\frac{2M}{A^2} - \frac{k^2}{A^2} = 0 \quad (21)$$

$$\Rightarrow m = \tfrac{1}{2}\left(-k\frac{2M}{A^2} \pm \sqrt{4k^2M^2A^{-4} + 4\frac{k^2}{A^2}}\right)$$

$$= \left(-k\frac{M}{A^2} \pm \frac{k}{A}\sqrt{\frac{M^2}{A^2} + 1}\right)$$

$$= \frac{k}{A}\left(-\frac{M}{A} \pm \sqrt{\frac{M^2}{A^2} + 1}\right). \quad (22)$$

Does this result make sense? Note as $M \to 0$ we recover $m = \pm \frac{k}{A}$.

In the hydrostatic limit $\frac{\omega^2}{N^2} \to 0$ and in the irrotational limit $\frac{f^2}{\omega^2} \to 0$ this further simplifies to $m = \pm \frac{k}{\sqrt{1 - 2i\frac{\alpha}{\omega} - \frac{\alpha^2}{\omega^2}}}$

$$= \pm \frac{k}{\left(1 - \frac{\alpha i}{\alpha}\right)} = \pm \frac{k\left(1 - i\frac{\alpha}{\omega}\right)}{\frac{\alpha^2}{\omega^2} + 1}.$$

In general it is possible to show the

roots of (21) are either 0, or of opposite sign, which we will need later!

To simplify the argument, first let $m = \frac{h}{A} n$. Then (21) becomes

$$\frac{h^2}{A^2} n^2 + \frac{h^2}{A} \frac{2M}{A^2} n - \frac{h^2}{A^2} = 0$$

$$\Rightarrow n^2 + \frac{2M}{A} n - 1 = 0 \quad (23).$$

Let n_1, n_2 be the roots of (23).

Vietas formulas imply $n_1 n_2 = -1$.

Let $P(n) = n^2 + \frac{2M}{A} n - 1$ and note $P'(c) = 0$

$$\Rightarrow 2c + \frac{2M}{A} = 0 \Rightarrow c = -\frac{M}{A}.$$

The Gauss-Lucas theorem therefore implies

$$\frac{1}{2}(n_1 + n_2) = c = -\frac{M}{A}. \quad (24)$$

Using $n_2 = -\frac{1}{n_1}$ we find

$2\frac{M}{A} = \left(\frac{1}{n_1} - n_1\right)$. (25) Let $n_1 = r_1 e^{i\theta_1}$, $A = r_A e^{i\theta_A}$.

Then (25) implies

$$\frac{2M}{r_A} e^{-i\theta_A} = \left(\frac{1}{r_1} e^{-i\theta_1} - r_1 e^{i\theta_1}\right)$$

$$\Rightarrow \frac{2M}{r_A} \cos(\theta_A) = \frac{1}{r_1} \cos(\theta_1) - r_1 \cos(\theta_1) \text{ and}$$

$$-\frac{2M}{r_A} \sin(\theta_A) = -\frac{1}{r_1} \sin(\theta_1) - r_1 \sin(\theta_1)$$

$$\Rightarrow \frac{2M}{r_A}\cos(\theta_A) = \left(\frac{1}{r_1} - r_1\right)\cos(\theta_1), \qquad (25)$$

$$\frac{2M}{r_A}\sin(\theta_A) = \left(\frac{1}{r_1} + r_1\right)\sin(\theta_1) \qquad (26).$$

Squaring and adding (25) and (26) gives

$$\left(\frac{2M}{r_A}\right)^2 = \left[\left(\frac{1}{r_1} - r_1\right)^2\cos^2(\theta_1) + \left(\frac{1}{r_1} + r_1\right)^2\sin^2(\theta_1)\right)$$

$$\Rightarrow \left(\frac{2M}{r_A}\right)^2 = \left(\frac{1}{r_1^2} - 2 + r_1^2\right)\cos^2\theta_1 + \left(\frac{1}{r_1^2} + 2 + r_1^2\right)\sin^2(\theta_1)$$

$$= \left(\frac{1}{r_1^2} + r_1^2\right) + 2\sin^2\theta_1 - 2\cos^2\theta_1$$

$$= \left(\frac{1}{r_1^2} + r_1^2\right) - 2\cos(2\theta_1)$$

$$\Rightarrow \left(\frac{2M}{r_A}\right)^2 + 2\cos(2\theta_1) = \frac{1}{r_1^2} + r_2^2 \qquad (27).$$

Squaring and subtracting (26) from (25) gives

$$\left(\frac{2M}{r_A}\right)^2\left(\cos^2(\theta_A) - \sin^2(\theta_A)\right)$$

$$= \left(\frac{1}{r_1^2} - 2 + r_1^2\right)\cos^2\theta_1 - \left(\frac{1}{r_1^2} + 2 + r_1^2\right)\sin^2\theta_1$$

$$\Rightarrow \left(\frac{2M}{r_A}\right)^2\cos(2\theta_A) = \left(\frac{1}{r_1^2} + r_1^2\right)\cos(2\theta_1) - 2 \qquad (28).$$

Substituting (27) into (28) gives

$$\left(\frac{2M}{r_A}\right)^2\cos(2\theta_A) = \left[\left(\frac{2M}{r_A}\right)^2 + 2\cos(2\theta_1)\right]\cos(2\theta_1) - 2$$

$$\Rightarrow \left(\frac{2M}{r_A}\right)^2\left[\cos(2\theta_A) - \cos(2\theta_1)\right] = 2\left[\cos^2(2\theta_1) - 1\right]$$

$$\Rightarrow \cos(2\theta_A) - \cos(2\theta_1) = \frac{-r_A^2}{2M^2}\sin^2(2\theta_1) \qquad (29).$$

Now, let $m_1 = \frac{h}{A}n_1$, $m_2 = \frac{h}{A}n_2$ be the roots of (21). Note that

$$\mathrm{Im}(m_1) = \mathrm{Im}\left(r_1 e^{i\theta_1} \frac{h}{r_A} e^{-i\theta_A}\right)$$
$$= r_1 \frac{h}{r_A} \sin(\theta_1 - \theta_A),$$
$$\mathrm{Im}(m_2) = \mathrm{Im}\left(\frac{-1}{r_1} e^{-i\theta_1} \frac{h}{r_A} e^{-i\theta_A}\right)$$
$$= \frac{h}{r_1 r_A} \sin(\pi - \theta_1 - \theta_A) = \frac{h}{r_1 r_A} \sin(\theta_1 + \theta_A).$$

So $\mathrm{Im}(m_1) \cdot \mathrm{Im}(m_2) = \frac{h^2}{r_A^2} \sin(\theta_1 - \theta_A) \sin(\theta_1 + \theta_A)$
$$= \frac{1}{2} \frac{h^2}{r_A^2} \left[\cos(2\theta_A) - \cos(2\theta_1)\right] \quad (30).$$

Substituting (29) into (30) we have
$$\mathrm{Im}(m_1) \cdot \mathrm{Im}(m_2) = -\frac{h^2}{2M^2} \sin^2(2\theta_1). \quad (31)$$

Thus $\mathrm{Im}(m_1) \cdot \mathrm{Im}(m_2) \leq 0$, and hence the imaginary parts of m_1, m_2 are either zero or of opposite sign!

We can ensure m_1 and m_2 are not purely real by ensuring A is not purely real or purely imaginary, which is the case when $\alpha > 0$.

Thus restricting to the $\alpha > 0$ case, our homogeneous equation
$$\hat{\psi}^h_{z_0 z_0} + ih \frac{2M}{A^2} \hat{\psi}^h_{z_0} + \frac{h^2}{A^2} \hat{\psi}^h = 0$$
has the general solution

$$\hat{\psi}^h = a_1 e^{im_1 z_\sigma} + a_2 e^{im_2 z_\sigma}, \quad \text{and}$$

because the imaginary parts of m_1 and m_2 are of opposite sign, one of the two terms above will decay with increasing z_σ, and one will grow; relabel these terms $a_n e^{im_n z_\sigma}$ and $a_p e^{im_p z_\sigma}$ accordingly, where $\operatorname{Im}(m_n) > 0$ and $\operatorname{Im}(m_p) < 0$.

We can now proceed much the same way as in previous work!

Note with the change to x_σ, z_σ coordinates, the lower boundary condition (6) can be re-written

$$w(z = \sigma(x)) = Mu(z = \sigma(x))$$

$$\Rightarrow -\psi_x(z = \sigma(x)) = M\psi_z(z = \sigma(x))$$

$$\Rightarrow -\left(\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right)\psi(z_\sigma = 0) = M\psi_{z_\sigma}(z_\sigma = 0)$$

$$\Rightarrow -\frac{\partial}{\partial x_\sigma}\psi(z_\sigma = 0) = 0$$

$$\Rightarrow -ik\hat{\psi}(z_\sigma = 0) = 0$$

$$\Rightarrow \hat{\psi}(z_\sigma = 0) = 0. \quad (32)$$

To solve for the Green's function G, we consider

$$G_{z_0 z_0} + i h \frac{2M}{A^2} G_{z_0} + \frac{h^2}{A^2} G = \delta(z_0 - z_0') . \quad (33)$$

For $z_0 < z_0'$ we have

$$G = a_n e^{i m_n z_0} + a_p e^{i m_p z_0} .$$

The lower boundary condition implies

$a_n + a_p = 0 \Rightarrow a_p = -a_n$ so

$$G = a_n \left(e^{i m_n z_0} - e^{i m_p z_0} \right) .$$

For $z_0 > z_0'$ we require that G decays as z_0 increases, so

$G = b_n e^{i m_n z_0}$. Continuity at z_0' gives

$$a_n \left(e^{i m_n z_0'} - e^{i m_p z_0'} \right) = b_n e^{i m_n z_0'} . \quad (34)$$

Also, (33) implies

$$\lim_{\epsilon \to 0} \int_{z_0' - \epsilon}^{z_0' + \epsilon} G_{z_0 z_0} + i h \frac{2M}{A^2} G_{z_0} + \frac{h^2}{A^2} G \, dz_0 = 1$$

$$\Rightarrow \lim_{z_0 \to z_0'^+} G_{z_0} - \lim_{z_0 \to z_0'^-} G_{z_0} = 1$$

$$\Rightarrow i m_n b_n e^{i m_n z_0'} - a_n \left(i m_n e^{i m_n z_0'} - i m_p e^{i m_p z_0'} \right) = 1$$

$$\Rightarrow i m_n b_n e^{i m_n z_0'} = 1 + i a_n \left(m_n e^{i m_n z_0'} - m_p e^{i m_p z_0'} \right) \quad (35)$$

Evaluating $(35) - i m_n (34)$ we have

$$= \frac{m_n - m_p}{m_n - m_p} = 1 \text{ as required!}$$

Thus $\hat{\Psi} = \int_0^\infty G(z_0, z_0') \frac{1}{2\bar{B}^2} \mathcal{F}\left\{\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right)H\right\}(z_0')\, dz_0'$

and $\Psi = \mathrm{Re}\left[\frac{1}{\pi}\int_0^\infty \hat{\Psi}\, e^{i(kx_0 + t)}\, dk\right]$.

Finally, we can translate this solution back to (x, z) coordinates.

Considering again equation (17), recall we also need to solve for the forcing $H(x,z)\frac{e^{-it}}{2}$. So we now assume

$\Psi = \tilde{\Psi}(x,z)e^{-it}$ and substitute into (17) to get

$$\left[\left(-1 - 2i\frac{\alpha}{\omega} + \left(\frac{\alpha}{\omega}\right)^2\right)\left(1 + M^2\frac{\omega^2}{N^2}\right) + \frac{\delta^2}{\omega^2} + M^2\right]\tilde{\Psi}_{z_0 z_0}$$
$$- \left[\frac{\omega^2}{N^2}\left(-1 - 2i\frac{\alpha}{\omega} + \frac{\alpha^2}{\omega^2}\right) + 1\right] 2M\,\tilde{\Psi}_{x_0 z_0}$$
$$+ \left[\frac{\omega^2}{N^2}\left(-1 - 2i\frac{\alpha}{\omega} + \frac{\alpha^2}{\omega^2}\right) + 1\right]\tilde{\Psi}_{x_0 x_0} = -\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right)\frac{H}{2}$$

$$\Rightarrow \left[\left(1 + 2i\frac{\alpha}{\omega} - \left(\frac{\alpha}{\omega}\right)^2\right)\left(1 + M^2\frac{\omega^2}{N^2}\right) - \frac{\delta^2}{\omega^2} - M^2\right]\hat{\Psi}_{z_0 z_0}$$
$$+ ik\left[\frac{\omega^2}{N^2}\left(-1 - 2i\frac{\alpha}{\omega} + \frac{\alpha^2}{\omega^2}\right) + 1\right] 2M\,\hat{\Psi}_{z_0}$$
$$+ k^2\left[\frac{\omega^2}{N^2}\left(-1 - 2i\frac{\alpha}{\omega} + \frac{\alpha^2}{\omega^2}\right) + 1\right]\hat{\Psi}$$
$$= \left(ik - M\frac{\partial}{\partial x_0}\right)\frac{\hat{H}}{2}$$

$$\Rightarrow \bar{B}^2\hat{\Psi}_{z_0 z_0} + ik\bar{C}^2 2M\hat{\Psi}_{z_0} + k^2\bar{C}^2\hat{\Psi} = \left(ik - M\frac{\partial}{\partial x_0}\right)\frac{\hat{H}}{2}$$

$$\Rightarrow \hat{\Psi}_{z_0 z_0} + \frac{ik2M}{\bar{A}^2}\hat{\Psi}_{z_0} + \frac{k^2}{\bar{A}^2}\hat{\Psi} = \frac{1}{2\bar{B}^2}\left(ik - M\frac{\partial}{\partial x_0}\right)\hat{H}.$$

The remainder of the argument is then as before, but with $\bar{A}, \bar{B}$ in place of A, B. Note that because $\alpha \neq 0$, we have

$$\frac{k}{\bar{A}}\left(-\frac{M}{\bar{A}} \pm \sqrt{\frac{M^2}{\bar{A}^2}+1}\right) = \overline{\frac{k}{A}\left(-\frac{M}{A} \pm \sqrt{\frac{M}{A}+1}\right)},$$

which implies in this case the vertical wavenumbers are $\overline{m_p}$, $\overline{m_n}$, noting that now $\operatorname{Im}(\overline{m_n}) < 0$ and $\operatorname{Im}(\overline{m_p}) > 0$. So G is now

$$G = \begin{cases} \dfrac{i}{(\overline{m_n}-\overline{m_p})} e^{-i\overline{m_n} z_0'}\left(e^{i\overline{m_n} z_0} - e^{i\overline{m_p} z_0}\right), & z_0 \leq z_0' \\ \dfrac{i}{(\overline{m_n}-\overline{m_p})}\left(e^{-i\overline{m_p} z_0'} - e^{-i\overline{m_n} z_0'}\right) e^{i\overline{m_p} z_0}, & z_0 > z_0'. \end{cases}$$

Note continuity clearly satisfied at z_0', and checking the "jump" condition,

$$G_{z_0}\big|_{z_0'^+} - G_{z_0}\big|_{z_0'^-}$$

$$= \frac{i}{(\overline{m_n}-\overline{m_p})} i\overline{m_p}\left(e^{-i\overline{m_p} z_0'} - e^{-i\overline{m_n} z_0'}\right) e^{i\overline{m_p} z_0'}$$

$$- \frac{i}{(\overline{m_n}-\overline{m_p})} i e^{-i\overline{m_n} z_0'}\left(\overline{m_n} e^{i\overline{m_n} z_0'} - \overline{m_p} e^{i\overline{m_p} z_0'}\right)$$

$$= \frac{-1}{(\overline{m_n}-\overline{m_p})} \overline{m_p}\left(1 - e^{i(\overline{m_p}-\overline{m_n}) z_0'}\right)$$

$$- \frac{-1}{\overline{m_n}-\overline{m_p}}\left(\overline{m_n} - \overline{m_p} e^{i(\overline{m_p}-\overline{m_n}) z_0'}\right)$$

$$= \frac{\overline{m_n}-\overline{m_p}}{\overline{m_n}-\overline{m_p}} = 1 \quad \text{as required.}$$

Now to write the complete solution,

let subscripts of p and n denote the solutions for ψ, $\tilde{\psi}$, $\hat{\psi}$ and G corresponding to the e^{it} and e^{-it} forcings, respectively. Then

$$\hat{\psi}_p = \int_0^\infty G_p(z_0, z_0') \frac{1}{2B^2} F\{(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0})H\}(z_0')\, dz_0' ,$$

$$\hat{\psi}_n = \int_0^\infty G_n(z_0, z_0') \frac{1}{2\bar{B}^2} F\{(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0})H\}(z_0')\, dz_0' ,$$

and

$$\psi_p = Re\left[\frac{1}{\pi}\int_0^\infty \hat{\psi}_p e^{i(hx_0+t)}\, dh\right],$$

$$\psi_n = Re\left[\frac{1}{\pi}\int_0^\infty \hat{\psi}_n e^{i(hx_0-t)}\, dh\right],$$

with $\psi = \psi_p + \psi_n$. For reference,

$$G_p = \begin{cases} \frac{i}{(m_n - m_p)} e^{-im_p z_0'}\left(e^{im_n z_0} - e^{im_p z_0}\right), & z_0 \le z_0' \\ \frac{i}{(m_n - m_p)}\left(e^{-im_p z_0'} - e^{-im_n z_0'}\right) e^{im_n z_0}, & z_0 > z_0' \end{cases},$$

$$G_n = \begin{cases} \frac{i}{(\bar{m}_n - \bar{m}_p)} e^{-i\bar{m}_n z_0'}\left(e^{i\bar{m}_n z_0} - e^{i\bar{m}_p z_0}\right), & z_0 \le z_0' \\ \frac{i}{(\bar{m}_n - \bar{m}_p)}\left(e^{-i\bar{m}_p z_0'} - e^{-i\bar{m}_n z_0'}\right) e^{i\bar{m}_p z_0}, & z_0 > z_0' \end{cases}.$$

Recall m_n, m_p are the roots of the quadratic $m^2 + h\frac{2M}{A^2}m - \frac{h^2}{A^2}$ chosen so that $\mathrm{Im}(m_n) > 0$ and $\mathrm{Im}(m_p) < 0$, and

$$B = \left[\left(1 - 2i\frac{\alpha}{\omega} - \frac{\alpha^2}{\omega^2}\right)\left(1 + M^2\frac{\omega^2}{N^2}\right) - \frac{\delta^2}{\omega^2} - M^2\right]^{1/2}$$

$$C = \left[1 + \frac{\omega^2}{N^2}\left(-1 + 2i\frac{\alpha}{\omega} + \frac{\alpha^2}{\omega^2}\right)\right]^{1/2}, \quad \text{with } A = \frac{B}{C}.$$

Point Forcing Solution

Lets try and find a purely analytic solution for the response to a point forcing. Let z_f be the height of the forcing (in z coordinates), with $z_f \geq Mx_f$, where x_f is the x coordinate of the forcing. Note we can define our coords such that $x_f = 0$. Then $H(x,z) = \delta(x)\,\delta(z - z_f)$.

In (x_0, z_0) coordinates we have

$$H(x_0, z_0) = \delta(x_0)\,\delta(z_0 + Mx_0 - z_f)$$
$$= \delta(x_0)\,\delta(z_0 - z_f).$$

Note the second line can be proved formally, but the intuition is that the $\delta(x_0)$ only "fires" when $x_0 = 0$, so we might as well set $x_0 = 0$ in $\delta(z_0 + Mx_0 - z_f)$. Now consider

$$\mathcal{F}\left\{\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right)H\right\} = ih\,\delta(z_0 - z_f) - M\,\delta'(z_0 - z_f).$$

$$\hat{\psi}_p = \int_0^\infty G_p(z_0, z_0')\frac{1}{2B^2}\,\mathcal{F}\left\{\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right)H\right\}\Big|_{z_0'}\,dz_0'$$

$$= \frac{1}{2B^2}\left[ik\, G_p(z_o, z_f) + M \frac{\partial G_p}{\partial z_o'}\bigg|_{z_f}\right].$$

Note

$$\frac{\partial G_p}{\partial z_o'} = \begin{cases} \dfrac{-i^2 m_p}{m_n - m_p}\, e^{-i m_p z_o'}\left(e^{i m_n z_o} - e^{i m_p z_o}\right), & z_o \le z_o' \\ \dfrac{-i^2}{m_n - m_p}\left(m_p e^{-i m_p z_o'} - m_n e^{-i m_n z_o'}\right) e^{i m_n z_o}, & z_o > z_o' \end{cases}$$

Now recall $\psi_p = \mathrm{Re}\left[\frac{1}{\pi}\int_0^\infty \hat{\psi}_p e^{i(kx_o - \omega t)}\, dk\right]$.

It looks like we'll be able to evaluate this analytically! From (22) we can write $m_n = k a_n$, $m_p = k a_p$, where $a_n, a_p \in \mathbb{C}$, and do not depend on k. Note k does not appear in B, or anywhere else in G_p or $\frac{\partial G_p}{\partial z_o'}$. Note first, if $z_o \le z_f$

$$\int_0^\infty k\, G_p(z_o, z_f)\, e^{ikx_o}\, dk$$

$$= \int_0^\infty \frac{i}{a_n - a_p} e^{-i a_p k z_f}\left(e^{i a_n k z_o} - e^{i a_p k z_o}\right) e^{ikx_o}\, dk$$

$$= \int_0^\infty \frac{i}{a_n - a_p}\left(e^{ik(a_n z_o - a_p z_f + x_o)} - e^{ik(a_p z_o - a_p z_f + x_o)}\right) dk$$

$$= \frac{1}{(a_n - a_p)}\left[\frac{1}{(a_n z_o - a_p z_f + x_o)} e^{ik(a_n z_o - a_p z_f + x_o)} - \frac{1}{(a_p(z_o - z_f) + x_o)} e^{ik(a_p(z_o - z_f) + x_o)}\right]_0^\infty .$$

For this integral to exist, we need exponential decay of both terms above as $k \to \infty$. But we know $\mathrm{Im}(m_n) > 0$ and

$\mathrm{Im}(w_p) < 0$, which implies

$\mathrm{Im}(a_n z_0) > 0$ and $\mathrm{Im}(-a_p z_f) > 0$,

if $z_0 > 0$ and $z_f > 0$. So in this case $e^{ik(a_n z_0 - a_p z_f + x_0)} \to 0$ as $k \to \infty$. But note we already know $\psi_p(z_0 = 0) = 0$ from the boundary condition, and so can ignore this case here.

Similarly, we know if $z_0 - z_f < 0$ we have $\mathrm{Im}(a_p(z_0 - z_f)) < 0$, so in this case we also have $e^{ik(a_p(z_0 - z_f) + x_0)} \to 0$ as $k \to \infty$; we ignore the $z_0 = z_f$ case for now. Thus for $0 < z_0 < z_f$,

$$\int_0^\infty \frac{i}{a_n - a_p} e^{-ia_p k z_f}\left(e^{ia_n k z_0} - e^{ia_p k z_0}\right) e^{ikx_0}\, dk$$

$$= \frac{1}{(a_n - a_p)}\left[\frac{1}{(a_p(z_0 - z_f) + x_0)} - \frac{1}{(a_n z_0 - a_p z_f + x_0)}\right].$$

Now, if $z_0 > z_f$, $\int_0^\infty k\, G_p(z_0, z_f)\, e^{ikx_0}\, dk$

$$= \int_0^\infty \frac{i}{(a_n - a_p)}\left(e^{-ia_p k z_f} - e^{-ia_n k z_f}\right) e^{ia_n k z_0} e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{i}{(a_n - a_p)}\left[e^{ik(a_n z_0 - a_p z_f + x_0)} - e^{ik(a_n(z_0 - z_f) + x_0)}\, dk\right]$$

$$= \frac{1}{(a_n - a_p)}\left[\frac{e^{ik(a_n z_0 - a_p z_f + x_0)}}{(a_n z_0 - a_p z_f + x_0)} - \frac{e^{ik(a_n(z_0 - z_f) + x_0)}}{(a_n(z_0 - z_f) + x_0)}\right]_0^\infty.$$

Note that each exponential term $\to 0$ as

$k \to \infty$ for analogous reasons to before. Thus for $z_0 > z_f$ we have

$$\int_0^\infty \frac{i}{(a_n - a_p)}\left[e^{-ia_pkz_f} - e^{-ia_nkz_f}\right] e^{ia_nkz_0} e^{ikx_0}\, dk$$

$$= \frac{1}{(a_n - a_p)}\left[\frac{1}{(a_n(z_0 - z_f) + x_0)} - \frac{1}{(a_n z_0 - a_p z_f + x_0)}\right].$$

For $z_0 = z_f$, continuity is achieved if the integral evaluates to

$$\frac{1}{(a_n - a_p)}\left[\frac{1}{x_0} - \frac{1}{((a_n - a_p)z_f + x_0)}\right],$$ which is consistent with both the above formulae.

Considering now $\int_0^\infty \frac{\partial G_p}{\partial z_0'}\Big|_{z_f} e^{ikx_0}\, dk$, for $0 < z_0 < z_f$ the calculation is similar to before, with

$$\int_0^\infty \frac{a_p}{a_n - a_p} e^{-ika_pz_f}\left(e^{ika_nz_0} - e^{ika_pz_0}\right) e^{ikx_0}\, dk$$

$$= \frac{a_p}{i(a_n - a_p)}\left[\frac{1}{(a_p(z_0 - z_f) + x_0} - \frac{1}{(a_n z_0 - a_p z_f + x_0)}\right].$$

For $z_0 > z_f$, we have

$$\int_0^\infty \frac{\partial G_p}{\partial z_0'}\Big|_{z_f} e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{1}{a_n - a_p}\left(a_p e^{-ia_pkz_f} - a_n e^{-ia_nkz_f}\right) e^{ika_nz_0} e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{1}{a_n - a_p}\left(a_p e^{ik(a_nz_0 - a_pz_f + x_0)} - a_n e^{ik(a_n(z_0 - z_f) + x_0)}\right) dk$$

$$= \frac{1}{i(a_n - a_p)}\left[\frac{a_p e^{ik(a_nz_0 - a_pz_f + x_0)}}{a_nz_0 - a_pz_f + x_0} - \frac{a_n e^{ik(a_n(z_0 - z_f) + x_0)}}{(a_n(z_0 - z_f) + x_0}\right]_0^\infty$$

$$\frac{1}{i(a_n - a_p)}\left[\frac{a_n}{a_n(z_0 - z_f) + x_0} - \frac{a_p}{(a_nz_0 - a_pz_f + x_0)}\right].$$

Now letting I_G^p and $I_{G'}^p$ denote these two integrals, we have

$$\psi_p = \mathrm{Re}\left[\frac{1}{2B^2}\left[i I_G^p + M I_{G'}^p\right]e^{it}\right].$$ Clearly the ψ_n case will be very similar. We have

$$\hat{\psi}_n = \int_0^\infty G_n(z_0, z_0')\frac{1}{2B^2}F\left\{\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right)H\right\}\Big|_{z_0'} dz_0'$$

$$= \frac{i}{2B^2}\left[ik\, G_n(z_0, z_f) + M\frac{\partial G_n}{\partial z_0'}\Big|_{z_f}\right].$$ Note

$$\frac{\partial G_n}{\partial z_0'} = \begin{cases} \dfrac{-i^2\bar{m}_n}{\bar{m}_n - \bar{m}_p} e^{-\bar{m}_n z_0'}\left(e^{i\bar{m}_n z_0} - e^{i\bar{m}_p z_0}\right), & z_0 \le z_0' \\ \dfrac{-i^2}{\bar{m}_n - \bar{m}_p}\left(\bar{m}_p e^{-i\bar{m}_p z_0'} - \bar{m}_n e^{-i\bar{m}_n z_0'}\right)e^{i\bar{m}_p z_0}, & z_0 > z_0'. \end{cases}$$

Now, for $0 < z_0 < z_f$,

$$I_G^n = \int_0^\infty k\, G_n(z_0, z_f)e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{i}{\bar{a}_n - \bar{a}_p} e^{-ik\bar{a}_n z_f}\left(e^{ik\bar{a}_n z_0} - e^{ik\bar{a}_p z_0}\right)e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{i}{\bar{a}_n - \bar{a}_p}\left(e^{ik(\bar{a}_n(z_0 - z_f) + x_0)} - e^{ik(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}\right)dk$$

$$= \frac{1}{\bar{a}_n - \bar{a}_p}\left[\frac{e^{ik(\bar{a}_n(z_0 - z_f) - z_f)}}{(\bar{a}_n(z_0 - z_f) + x_0} - \frac{e^{ik(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}}{(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}\right]_0^\infty$$

$$= \frac{1}{\bar{a}_n - \bar{a}_p}\left[\frac{1}{\bar{a}_p z_0 - \bar{a}_n z_f + x_0} - \frac{1}{(\bar{a}_n(z_0 - z_f) + x_0)}\right].$$

For $z_0 > z_f$,

$$I_G^n = \int_0^\infty \frac{i}{\bar{a}_n - \bar{a}_p}\left(e^{-ik\bar{a}_p z_f} - e^{-ik\bar{a}_n z_f}\right)e^{ik\bar{a}_p z_0} e^{ikx_0}\, dk$$

$$= \int_0^\infty \frac{i}{\bar{a}_n - \bar{a}_p}\left(e^{ik(\bar{a}_p(z_0 - z_f) + x_0)} - e^{ik(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}\right)dk$$

$$= \frac{1}{\bar{a}_n - \bar{a}_p}\left[\frac{e^{ik(\bar{a}_p(z_0 - z_f) + x_0)}}{(\bar{a}_p(z_0 - z_f) + x_0)} - \frac{e^{ik(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}}{(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}\right]_0^\infty$$

$$= \frac{1}{\bar{a}_n - \bar{a}_p}\left[\frac{1}{\bar{a}_p z_0 - \bar{a}_n z_f + x_0} - \frac{1}{\bar{a}_p(z_0 - z_f) + x_0}\right]$$

Considering now $I_{a'}^{n}$, for $0 < z_0 < z_f$ we have

$$\int_0^\infty \frac{\partial h_n}{\partial z_0'}\Big|_{z_f} e^{ikx_0}\,dk$$

$$= \int_0^\infty \frac{\bar{a}_n}{\bar{a}_n - \bar{a}_p} e^{-\bar{a}_n k z_f}\left(e^{i\bar{a}_n k z_0} - e^{i\bar{a}_n k z_0}\right) e^{ikx_0}\,dk$$

$$= \frac{\bar{a}_n}{i(\bar{a}_n - \bar{a}_p)}\left[\frac{1}{(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)} - \frac{1}{(\bar{a}_n(z_0 - z_f) + x_0)}\right].$$

For $z_0 > z_f$ we have $\int_0^\infty \frac{\partial h_n}{\partial z_0'}\Big|_{z_f} e^{ikx_0}\,dk$

$$\int_0^\infty \frac{1}{(\bar{a}_n - \bar{a}_p)}\left(\bar{a}_p e^{ik(\bar{a}_p(z_0 - z_f) + x_0)} - \bar{a}_n e^{ik(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)}\right) dk$$

$$= \frac{1}{i(\bar{a}_n - \bar{a}_p)}\left[\frac{\bar{a}_n}{(\bar{a}_p z_0 - \bar{a}_n z_f + x_0)} - \frac{\bar{a}_p}{(\bar{a}_p(z_0 - z_f) + x_0)}\right].$$

Now recall $\Psi_n = \mathrm{Re}\left[\frac{1}{2\bar{B}^2}\left[i I_h^n + M I_{a'}^n\right] e^{-it}\right].$

Exponential Heating Solution

Consider now $H(x,z) = e^{-(z-\sigma(x))} L(x)$ for some function L. In x_σ, z_σ coordinates we have $H(x_\sigma, z_\sigma) = e^{-z_\sigma} L(x_\sigma)$ and

$$F\left\{\left(\frac{\partial}{\partial x_\sigma} - M\frac{\partial}{\partial z_\sigma}\right)H\right\} = (ik+M)\,e^{-z_\sigma}\,\hat{L}(x_\sigma).$$

Then $\hat{\psi}_p = \int_0^\infty G_p(z_\sigma, z_\sigma')\dfrac{(ik+M)\hat{L}}{2B^2} e^{-z_\sigma'}\,dz_\sigma'$

$$= \frac{i(ik+M)\hat{L}}{2B^2(m_n-m_p)}\left\{\int_0^{z_\sigma}\left(e^{(-im_p-1)z_\sigma'} - e^{(-im_n-1)z_\sigma'}\right)e^{im_n z_\sigma}\,dz_\sigma' + \int_{z_\sigma}^\infty e^{(-im_p-1)z_\sigma'}\left(e^{im_n z_\sigma} - e^{im_p z_\sigma}\right)dz_\sigma'\right\}$$

$$= \frac{i(ik+M)\hat{L}}{2B^2(m_n-m_p)}\left\{\left[\frac{e^{(-im_p-1)z_\sigma'}}{(-im_p-1)} - \frac{e^{(-im_n-1)z_\sigma'}}{(-im_n-1)}\right]_0^{z_\sigma} e^{im_n z_\sigma} + \left[\frac{e^{(-im_p-1)z_\sigma'}}{(-im_p-1)}\right]_{z_\sigma}^\infty \left(e^{im_n z_\sigma} - e^{im_p z_\sigma}\right)\right\}$$

$$= \frac{i(ik+M)\hat{L}}{2B^2(m_n-m_p)}\left\{\left[\frac{1}{(-im_p-1)}\left(e^{(-im_p-1)z_\sigma} - 1\right) - \frac{1}{(-im_n-1)}\left(e^{(-im_n-1)z_\sigma} - 1\right)\right] \times e^{im_n z_\sigma} - \frac{1}{(-im_p-1)}e^{(-im_p-1)z_\sigma}\left(e^{im_n z_\sigma} - e^{im_p z_\sigma}\right)\right\}$$

cancels!

$$= \frac{i(ik+M)\hat{L}}{2B^2(m_n-m_p)}\left\{\frac{-e^{im_n z_\sigma}}{(-im_p-1)} - \frac{1}{(-im_n-1)}\left(e^{-z_\sigma} - e^{im_n z_\sigma}\right) + \frac{1}{(-im_p-1)}e^{-z_\sigma}\right\}$$

$$= \frac{i(ik+M)\hat{L}}{2B^2(m_n-m_p)}\left\{\left(\frac{1}{(-im_p-1)} - \frac{1}{(-im_n-1)}\right)\left(e^{-z_\sigma} - e^{im_n z_\sigma}\right)\right\}.$$

Similarly we have $\hat{\psi}_n = \int_0^\infty G_n(z_\sigma, z_\sigma')\dfrac{(ik+M)\hat{L}}{2\bar{B}^2} e^{-z_\sigma'}\,dz_\sigma'$

$$= \frac{i(ik+M)\hat{L}}{2\bar{B}^2(\bar{m}_n-\bar{m}_p)}\left\{\int_0^{z_\sigma}\left(e^{(-i\bar{m}_p-1)z_\sigma'} - e^{(-i\bar{m}_n-1)z_\sigma'}\right)e^{i\bar{m}_p z_\sigma}\,dz_\sigma' + \int_{z_\sigma}^\infty e^{(-i\bar{m}_n-1)z_\sigma'}\left(e^{i\bar{m}_n z_\sigma} - e^{i\bar{m}_p z_\sigma}\right)dz_\sigma'\right\}$$

$$= \frac{i(ik+M)\hat{L}}{2\bar{B}^2(\bar{m}_n-\bar{m}_p)}\left\{\left[\frac{1}{(-i\bar{m}_p-1)}\left(e^{(-i\bar{m}_p-1)z_\sigma} - 1\right) - \frac{1}{(-i\bar{m}_n-1)}\left(e^{(-i\bar{m}_n-1)z_\sigma} - 1\right)\right]e^{i\bar{m}_p z_\sigma} - \frac{1}{(-i\bar{m}_n-1)}e^{(-i\bar{m}_n-1)z_\sigma}\left(e^{i\bar{m}_n z_\sigma} - e^{i\bar{m}_p z_\sigma}\right)\right\}$$

cancels!

$$= \frac{i(ik+M)\hat{L}}{2\bar{B}^2(\bar{m}_n-\bar{m}_p)}\Big\{\frac{1}{(-i\bar{m}_p-1)}\left(e^{-z_0}-e^{i\bar{m}_p z_0}\right)$$
$$+\frac{1}{(-i\bar{m}_n-1)}e^{i\bar{m}_p z_0}-\frac{1}{(-i\bar{m}_n-1)}e^{-z_0}\Big\}$$
$$= \frac{i(ik+M)\hat{L}}{2\bar{B}^2(\bar{m}_n-\bar{m}_p)}\left\{\left(\frac{1}{(-i\bar{m}_p-1)}-\frac{1}{(-i\bar{m}_n-1)}\right)\left(e^{-z_0}-e^{i\bar{m}_p z_0}\right)\right\}.$$

Hmm, we can simplify $\hat{\psi}_p$ and $\hat{\psi}_n$ slightly by noting $\frac{1}{(m_n-m_p)}\left[\frac{1}{(-im_p-1)}-\frac{1}{(-im_n-1)}\right]$

$$= \frac{1}{m_n-m_p}\left[\frac{(-im_n-1)-(-im_p-1)}{(-im_p-1)(-im_n-1)}\right] = \frac{1}{m_n-m_p}\left[\frac{i(m_p-m_n)}{(-im_p-1)(-im_n-1)}\right]$$

$= \frac{-i}{(-im_p-1)(-im_n-1)}$, and similarly,

$\frac{1}{(\bar{m}_n-\bar{m}_p)}\left[\frac{1}{(-i\bar{m}_p-1)}-\frac{1}{(-i\bar{m}_n-1)}\right] = \frac{-i}{(-i\bar{m}_p-1)(-i\bar{m}_n-1)}$. So

$$\hat{\psi}_p = \frac{(ik+M)\hat{L}}{2B^2(im_p+1)(im_n+1)}\left(e^{-z_0}-e^{im_n z_0}\right),$$
$$\hat{\psi}_n = \frac{(ik+M)\hat{L}}{2\bar{B}^2(i\bar{m}_p+1)(i\bar{m}_n+1)}\left(e^{-z_0}-e^{i\bar{m}_p z_0}\right)$$

Mountain-Valley Breeze

Now, from our exponential heating solution, if we simply take $L(x) = 1$, we have $F[L] =: \hat{L} = 2\pi\,\delta(k)$. Then

$$\hat{\psi}_p = \frac{\pi M \delta(k)}{B^2}\left(e^{-z_0} - 1\right),$$

$$\hat{\psi}_n = \frac{\pi M \delta(k)}{\bar{B}^2}\left(e^{-z_0} - 1\right).$$ Writing

$B = |B| e^{i\arg(B)}$ we have

$$\frac{1}{B^2} = \frac{1}{|B|^2} e^{-2i\arg(B)}, \quad \frac{1}{\bar{B}^2} = \frac{1}{|B|^2} e^{2i\arg(B)},$$

$$\psi_p = \mathrm{Re}\left[\frac{M}{2|B|^2}\left(e^{-z_0} - 1\right) e^{i(t - 2\arg(B))}\right],$$

$$\psi_n = \mathrm{Re}\left[\frac{M}{2|B|^2}\left(e^{-z_0} - 1\right) e^{-i(t - 2\arg(B))}\right],$$ so

$$\psi = \frac{M}{|B|^2}\left(e^{-z_0} - 1\right)\cos\left(t - 2\arg(B)\right).$$ Note

we can of course also produce this result more directly. Lets go all the way back to equation (17). We have

$$\left[\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)^2\left(1 + M^2\frac{\omega^2}{N^2}\right) + \frac{f^2}{\omega^2} + M^2\right]\psi_{z_0 z_0}$$

$$- \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)^2 + 1\right] 2M\,\psi_{x_0 z_0} + \left[\frac{\omega^2}{\omega^2}\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)^2 + 1\right]\psi_{x_0 x_0}$$

17. $$= -\left(\frac{\partial}{\partial x_0} - M\frac{\partial}{\partial z_0}\right) Q$$

Writing $Q = \mathrm{Re}[e^{z_0} e^{it}]$ and

$\psi = \mathrm{Re}[\tilde{\psi} e^{it}]$ and putting into (17) gives

$$-B^2 \tilde{\psi}_{z_0 z_0} - c^2 2M \tilde{\psi}_{x_0 z_0} + c^2 \tilde{\psi}_{x_0 x_0} = -M e^{-z_0}$$

$$\Rightarrow \tilde{\psi}_{z_0 z_0} + \frac{2M}{A^2} \tilde{\psi}_{x_0 z_0} - \frac{1}{A^2} \tilde{\psi}_{x_0 x_0} = \frac{M}{B^2} e^{-z_0}$$

$$= \frac{M}{|B|^2} e^{-z_0} e^{-2i\arg(B)} . \quad (39).$$

Clearly $\tilde{\psi} = \frac{M}{|B|^2}\left(e^{-z_0} - 1\right) e^{-2i\arg(B)}$ solves

(39), so $\psi = \mathrm{Re}\left[\frac{M}{|B|^2}\left(e^{-z_0} - 1\right) e^{i(t - 2\arg(B))}\right]$

$= \frac{M}{|B|^2}\left(e^{-(z-Mx)} - 1\right) \cos(t - 2\arg(B))$

matching the expression using the more general method from before!

Now $u = \frac{\partial \psi}{\partial z} = \frac{-M}{|B|^2} e^{-(z-Mx)} \cos(t - 2\arg(B))$,

$w = -\frac{\partial \psi}{\partial x} = \frac{-M^2}{|B|^2} e^{-(z-Mx)} \cos(t - 2\arg(B))$,

and because $M > 0$ upslope winds correspond to $u > 0$, $w > 0$, and these therefore peak when

$\cos(t_{Mw} - 2\arg(B)) = -1$

$\Rightarrow t_{Mw} - 2\arg(B) = \pi(2j+1)\,, \quad j \in \mathbb{Z}.$

$\Rightarrow t_{Mw} = (2j+1)\pi + 2\arg(B)\,, \quad j \in \mathbb{Z}.$

But the heating $Q \sim \cos(t)$ clearly peaks at $t_{MQ} = 2l\pi$, $l \in \mathbb{Z}$, so $\Delta t = \pi + 2\arg(B)$. Notice that for inviscid, hydrostatic cases we have $B = \sqrt{1 - \frac{\delta^2}{\omega^2} - M^2}$, and so when $1 - \frac{\delta^2}{\omega^2} - M^2 > 0$, $\arg(B) = 0$, and the winds and heating are a half-cycle out of phase! If $1 - \frac{\delta^2}{\omega^2} - M^2 < 0$ we have $B = i \cdot \sqrt{M^2 + \frac{\delta^2}{\omega^2} - 1}$, and so $\arg(B) = \frac{\pi}{2}$, giving $\Delta t = 2\pi$, bringing the heating and up-slope winds back into phase! This is the mountain-valley breeze analogue to the celebrated Rotunno (1983) result for the land-sea breeze!