Ewan Short 15/9/25

Rotunno (1983) Problem

Now consider $Q = e^{-z} e^{it} \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{L}\right)$.

The idea here is that we will now transition smoothly from the "sea" region (no heating) to the "land" region (heating) over a "coastal zone". The width of the coastal zone is governed by L. We recover the heaviside problem as $L \to 0$. Note

$$\tilde{Q} = e^{-z} 2\pi \delta(\sigma-1) \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{L}\right),$$

$$\frac{\partial \tilde{Q}}{\partial x} = e^{-z} 2\pi \delta(\sigma-1) \frac{1}{\pi}\left(\frac{1}{\left(\frac{x}{L}\right)^2 + 1}\right) \cdot \frac{1}{L}$$

$$\widehat{\frac{\partial Q}{\partial x}} = e^{-z} 2\pi \delta(\sigma-1) e^{-L|k|}.$$

So the only change from the Heaviside problem is the factor $e^{-L|k|}$. So (22) from the Heaviside problem is now

$$\tilde{\psi} = \frac{-\delta(\sigma-1)}{B^2 2i}\left\{\int_0^\infty\left(\frac{-1}{(k/A+i)}+\frac{1}{(k/A-i)}\right)\left(e^{ik/A z}-e^{-z}\right)e^{ikx}e^{-Lk}\,dk\right.$$

$$\left.+\int_{-\infty}^0\left(\frac{-1}{(k/A+i)}+\frac{1}{(k/A-i)}\right)\left(e^{-ik/A z}-e^{-z}\right)e^{ikx}e^{Lk}\,dk\right\},$$

recalling we choose $\frac{1}{A}$ so that $\Im(1/A)>0$.

So define

$$L_1 = 1/A\,z + x + iL, \qquad L_2 = -1/A\,z + x - iL.$$

Note $\operatorname{Im}(L_1) = \operatorname{Im}(1/A\,z + x + iL)$

$= (\operatorname{Im}(1/A) + L)\operatorname{sgn}(z) \geq 0$, noting $z \geq 0$

and $L>0$. Also, note $\arg(-iL_1)$

$= \operatorname{atan2}(\operatorname{Im}(-iL_1), \operatorname{Re}(-iL_1))$

$= \operatorname{atan2}(-\operatorname{Re}(1/A)z - x, \operatorname{Im}(1/A)z + L)$,

so $\arg(-iL_1) \in (-\pi/2, \pi/2)$ $\forall z>0$, noting

$\operatorname{Im}(1/A)>0$, $L>0$. So just like in

the Heaviside problem we find

$$\int_0^\infty \frac{-1}{(k/A+i)}e^{ikL_1}\,dk = -Ae^{AL_1}\left[-i\pi - E_i^{\pi/2}(-AL_1)\right].$$

Similarly, we find

$$\int_0^\infty \frac{1}{(k/A+i)}e^{-z}e^{ik(x+iL)}\,dk = Ae^{A(x+iL)}e^{-z}\left(-i\pi - E_i^{\pi/2}(-A(x+iL))\right)$$

$$\int_0^\infty \frac{1}{h/A - i} e^{ihL_1}\, dh = Ae^{-AL_1}\left(i\pi - Ei^{\pi}(AL_1)\right)$$

$$\int_0^\infty \frac{-1}{h/A - i} e^{-z} e^{ih(x+iL)} = -Ae^{-A(x+iL)}\left(i\pi - Ei^{\pi}(A(x+iL))\right)e^{-z}$$

$$\int_{-\infty}^0 \frac{-1}{h/A + i} e^{ihL_2}\, dh = -Ae^{AL_2}\left(Ei^{\pi}(-AL_2) + i\pi\right)$$

$$\int_{-\infty}^0 \frac{1}{h/A + i} e^{-z} e^{ih(x-iL)}\, dh = Ae^{A(x-iL)} e^{-z}\left(Ei^{\pi}(-A(x-iL)) + i\pi\right)$$

$$\int_{-\infty}^0 \frac{1}{(h/A - i)} e^{ihL_2}\, dh = Ae^{-AL_2}\left(Ei^{\pi/2}(AL_2) + i\pi\right)$$

$$\int_{-\infty}^0 \frac{-1}{(h/A - i)} e^{-z} e^{ih(x-iL)} = -Ae^{-A(x-iL)}\left(Ei^{\pi/2}(A(x-iL)) + i\pi\right)e^{-z}.$$

Label these integrals $I_1, I_2, \ldots, I_8$, respectively. Then

$$\psi = \mathrm{Re}\left\{\frac{-1}{2\pi B^2 2i} I e^{it}\right\} \quad \text{where} \quad I = \sum_{n=1}^{\infty} I_n.$$

To get u, note

$$\frac{\partial I_1}{\partial z} = I_1 - Ae^{AL_1}\left(-\frac{e^{-AL_1}}{-AL_1}\cdot(-1)\right) = I_1 + \frac{1}{L_1}.$$

$$\frac{\partial I_2}{\partial z} = -I_2$$

$$\frac{\partial I_3}{\partial z} = -I_3 - A\frac{e^{-AL_1}\ e^{AL_1}}{AL_1} = -\frac{1}{L_1} - I_3$$

$$\frac{\partial I_4}{\partial z} = -I_4$$

$$\frac{\partial I_5}{\partial z} = -I_5 - Ae^{AL_2}\left(\frac{e^{-AL_2}}{-AL_2}\right)\cdot 1 = -I_5 + \frac{1}{L_2}$$

$$\frac{\partial I_6}{\partial z} = -I_6$$

$$\frac{\partial I_7}{\partial z} = I_7 + Ae^{-AL_2}\frac{e^{AL_2}}{AL_2}\cdot(-1) = I_7 - \frac{1}{L_2}$$

$\dfrac{\partial I_8}{\partial z} = -I_8$. So for u we have

$u = \mathrm{Re}\left\{\dfrac{-1}{2\pi\beta^2 2i} I_u e^{it}\right\}$, where

$$I_u = I_1 - I_2 - I_3 - I_4 - I_5 - I_6 + I_7 - I_8.$$

Similarly,

$$\frac{\partial I_1}{\partial x} = AI_1 - Ae^{AL_1}\left[-\frac{e^{-AL_1}}{-AL_1}\right]\cdot(-A) = AI_1 + \frac{A}{L_1}$$

$$\frac{\partial I_2}{\partial x} = AI_2 + Ae^{A(x+iL)}e^{-z}\left(-\frac{e^{-A(x+iL)}}{-A(x+iL)}\right)\cdot(-A)$$

$$= AI_2 + \frac{A\cdot(-A)e^{-z}}{A(x+iL)} = AI_2 - \frac{Ae^{-z}}{(x+iL)}$$

$$\frac{\partial I_3}{\partial x} = -I_3A + Ae^{-AL_1}\left(-\frac{e^{AL_1}}{AL_1}\right)\cdot A = -AI_3 - \frac{A}{L_1}$$

$$\frac{\partial I_4}{\partial x} = -AI_4 + Ae^{-A(x+iL)}\left(\frac{e^{A(x+iL)}}{A(x+iL)}\right)\cdot Ae^{-z}$$

$$= -AI_4 + \frac{Ae^{-z}}{x+iL}$$

$$\frac{\partial I_5}{\partial x} = AI_5 - Ae^{AL_2}\left(\frac{e^{-AL_2}}{-AL_2}\right)(-A) = AI_5 - \frac{A}{L_2}$$

$$\frac{\partial I_6}{\partial x} = AI_6 + Ae^{A(x-iL)}e^{-z}\left(\frac{e^{-A(x-iL)}}{-A(x-iL)}\right)(-A)$$

$$= AI_6 + \frac{Ae^{-z}}{x-iL}$$

$$\frac{\partial I_7}{\partial x} = -AI_7 + Ae^{-AL_2}\left(\frac{e^{AL_2}}{AL_2}\right)\cdot A = -AI_7 + \frac{A}{L_2}$$

$$\frac{\partial I_8}{\partial x} = -AI_8 - Ae^{-A(x-iL)}\left(\frac{e^{A(x-iL)}}{A(x-iL)}\right)\cdot Ae^{-z} = -AI_8 - \frac{Ae^{-z}}{x-iL}.$$

Now recalling $w = -\frac{\partial \psi}{\partial x}$, we have

$$w = \mathrm{Re}\left\{ \frac{-1}{2\pi \beta^2 \lambda \sigma} I_w e^{\sigma t} \right\}$$ where

$$I_w = A\left(-I_1 - I_2 + I_3 + I_4 - I_5 - I_6 + I_7 + I_8\right).$$