Ewan Short 2/9/25

Exponential Heating

Now let's consider $Q = e^{-z} H(x,t)$ for some function H. Then, defining $\chi(z) = \operatorname{sgn}(\operatorname{Im}(z))$ for $z \in \mathbb{C}$,

$$\hat{\psi} = \int_0^\infty \frac{ik}{B^2} \hat{Q}(z') G(z,z')\, dz'$$

$$= \frac{-ik\hat{H}}{B^2 m} \Big\{ e^{\chi(m) i m z} \int_0^z e^{-z'} \sin(mz')\, dz'$$

$$+ \sin(mz) \int_z^\infty e^{-z'} e^{\chi(m) i m z'}\, dz' \Big\}.$$

Note first

$$\int_0^z e^{-z'} \sin(mz')\, dz' = \int_0^z e^{-z'} \frac{e^{imz'} - e^{-imz'}}{2i}\, dz'$$

$$= \left[\frac{1}{2i} \left(\frac{1}{im-1} e^{(im-1)z'} - \frac{1}{(-im-1)} e^{(-im-1)z'} \right) \right]_0^z$$

$$= \frac{1}{2i} \left[\frac{1}{(im-1)} \left(e^{(im-1)z} - 1 \right) - \frac{1}{(-im-1)} \left(e^{(-im-1)z} - 1 \right) \right]$$

$$= \frac{1}{2i(m^2+1)} \left[(-im-1)\left(e^{(im-1)z} - 1 \right) - (im-1)\left(e^{(-im-1)z} - 1 \right) \right]$$

$$= \frac{1}{2i(m^2+1)} \Big[e^{-z} \left(-im e^{imz} - im e^{-imz} - e^{imz} + e^{-imz} \right)$$

$$- (-im-1) + (im-1) \Big]$$

$$= \frac{1}{(m^2+1)} \left[e^{-z} \left(-m\cos(mz) - \sin(mz) \right) + m \right].$$

Also,

$$\int_z^\infty e^{-z'} e^{\chi(m) i m z'}\, dz' = \frac{1}{(\chi(m) i m - 1)} \left[e^{(\chi(m) i m - 1) z'} \right]_z^\infty$$

$$= \frac{(-\chi(m) i m - 1)}{(m^2+1)} \left(0 - e^{(\chi(m) i m - 1) z}\right)$$

$$= \frac{\chi(m) i m + 1}{m^2+1} e^{(\chi(m) i m - 1) z} . \quad \text{So}$$

$$\hat{\psi} = \frac{-i A \hat{H}}{B^2 (m^2+1)} \Big\{ e^{(\chi(m) i m - 1) z} \left(-m \cos(mz) - \sin(mz)\right)$$

$$+ m e^{\chi(m) i m z} + \sin(mz) \left(\chi(m) i m + 1\right) e^{(\chi(m) i m - 1) z} \Big\}$$

cancels

$$= \frac{-i A \hat{H}}{B^2 (m^2+1)} \Big\{ e^{(\chi(m) i m - 1) z} \big(\underbrace{-m \cos(mz) + \sin(mz) \chi(m) i m}_{= - e^{-\chi(m) i m z}}\big)$$

$$+ m e^{\chi(m) i m z} \Big\}$$

$$= \frac{-i A \hat{H}}{B^2 (m^2+1)} \Big\{ m e^{\chi(m) i m z} - m e^{-z} \Big\} .$$

$$= \frac{-i m A \hat{H}}{B^2 (m^2+1)} \Big\{ e^{\chi(m) i m z} - e^{-z} \Big\} .$$

Note the consistency of this expression with Qian (2009), Rotunno (1983) etc.

Ewan Short 2/9/25

"Sea-Breeze" Forcing

Now let's try for a purely analytic solution for an idealized land-sea breeze type forcing. Lets try

Note it is easier for this problem to make the forcing $\in \mathbb{C}$ then take the real part of ψ at the end noting now $\psi = \mathrm{Re}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi} e^{i\sigma t} d\sigma\}$

$$Q = e^{-z}\,\theta(x)\,e^{it}$$

$$\tilde{Q} = e^{-z}\,\theta(x)\,2\pi\,\delta(\sigma-1) \text{ and}$$

$$ik\hat{Q} = e^{-z}\,2\pi\,\delta(\sigma-1)\,\widehat{\frac{\partial\theta}{\partial x}} = e^{-z}\,2\pi\,\delta(\sigma-1)\,\widehat{\delta(x)}$$

$= e^{-z}\,2\pi\,\delta(\sigma-1)$. So with only trivial changes to our previous algebra, we find

$$\hat{\psi} = \frac{-2\pi\,\delta(\sigma-1)}{B^2\left(\frac{k^2}{A^2}+1\right)}\left\{e^{\mathrm{sgn}(k)\mathcal{X}(1/A)\,i\frac{k}{A}z} - e^{-z}\right\}.$$

But this ~~should~~ means $\tilde{\psi}$ should have solutions using the exponential integral after a partial fractions decomposition. Note

$$\left(\frac{k^2}{A^2}+1\right) = \frac{1}{A^2}\left(k^2 + A^2\right) = \frac{1}{A^2}(k+iA)(k-iA).$$

Also,
$$\frac{1}{(k+iA)(k-iA)} = \frac{N_1}{k+iA} + \frac{N_2}{k-iA}$$

$\Rightarrow 1 = N_1(k - iA) + N_2(k + iA)$

$\Rightarrow 1 = N_2 \, 2iA \;\wedge\; 1 = -N_1 \, 2iA$

$\Rightarrow N_2 = \frac{1}{2iA} \;\wedge\; N_1 = {}^{-1}/_{2iA}$. So

$$\tilde{\Psi} = \frac{-\delta(\sigma-1)}{B^2 \, {}^1/_{A^2} \, 2iA} \int_{-\infty}^{\infty} \left(\frac{-1}{k+iA} + \frac{1}{k-iA}\right)\left(e^{\mathrm{sgn}(k) x \delta ({}^1/_A) i {}^k/_A z} - e^{-z}\right) e^{ikx} \, dk$$

$$= \frac{-\delta(\sigma-1)}{B^2 \, 2i} \Big\{ \int_0^{\infty} \left(\frac{-1}{({}^k/_A + i)} + \frac{1}{({}^k/_A - i)}\right)\left(e^{x({}^1/_A) i {}^k/_A z} - e^{-z}\right) e^{ikx} \, dk$$

$$\int_{-\infty}^{0} \left(\frac{-1}{({}^k/_A + i)} + \frac{1}{({}^k/_A - i)}\right)\left(e^{-x({}^1/_A) i {}^k/_A z} - e^{-z}\right) e^{ikx} \, dk \Big\}. \quad (22)$$

To evaluate these integrals analytically, we need anti-derivatives to expressions of the form $\frac{1}{{}^k/_A + a} e^{ikb}$, for $a, b \in \mathbb{C}$.

Define $u = ib(aA + k)$ so that $\frac{u}{ibA} = ({}^k/_A + a)$ and $\frac{\partial u}{\partial k} = ib$. But note $\frac{1}{{}^k/_A + a} e^{ikb}$

$$= \frac{ibA}{u} e^{u} \cdot e^{-iabA} = \frac{\partial}{\partial u}\left[ibA \cdot e^{-iabA} \int_{-\infty}^{u} \frac{e^{t}}{t} \, dt\right]$$

$$= \frac{\partial}{\partial k}\left[A e^{-iabA} \, \mathrm{Ei}(ib(aA + k))\right].$$ So

$A e^{-iabA} \mathrm{Ei}(ib(aA + k))$ is an anti-derivative of $\frac{1}{{}^k/_A + a} e^{ikb}$. Now, to calculate the values of these anti-derivatives as

$h \to \pm\infty$, we need some limit results about Ei. Recall $Ei(z) = \int_{-\infty}^{z} \frac{e^{t}}{t}\,dt$ and $E_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t}\,dt$. Typically Ei and E_1 are defined using branch cuts along the negative real axis; here we will suppose the functions have arbitrary branch cuts along the line $z_b = re^{i\theta_b}$, with $r \in (0,\infty)$ and $\theta_b \in (-\pi,\pi]$, with the same branch cut defined for $\ln$.

Consider $z = re^{i\theta} \in \mathbb{C}$, with $\theta \neq \theta_b$ and $\theta \in (\theta_b - \pi, \theta_b) \cup (\theta_b, \theta_b + \pi)$.

If $\theta > \theta_b$ then $-z = re^{i(\theta-\pi)}$, if $\theta < \theta_b$ then $-z = re^{i(\theta+\pi)}$. So if $\theta > \theta_b$

$Ei(-z) + E_1(z) = \ln(re^{i(\theta-\pi)}) - \ln(re^{i\theta})$
$= \ln(r) + i\theta - i\pi - \ln(r) - i\theta = -i\pi$. Similarly, if $\theta < \theta_b$, $Ei(-z) + E_1(z) = i\pi$.

Taking $\arg : \mathbb{C} \to [\theta_b - \pi, \theta_b) \cup [\theta_b, \theta_b + \pi)$

So $Ei(-z) + E_1(z) = \begin{cases} -i\pi & \text{if } \arg(z) > \theta_b, \\ i\pi & \text{if } \arg(z) < \theta_b, \end{cases}$

$\Leftrightarrow Ei(z) + E_1(-z) = \begin{cases} i\pi & \text{if } \arg(-z) > \theta_b, \\ -i\pi & \text{if } \arg(-z) < \theta_b. \end{cases}$

Now, note that $E_1(-z) = \int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt$
$= \int_1^{\infty} \frac{e^{sz}}{-sz}(-z) \, ds = \int_1^{\infty} \frac{e^{sz}}{s} \, ds$, where we have substituted $t = -sz$ with $s \in \mathbb{R}$. Now, note that $E_1(-ib(aA+k)) = \int_1^{\infty} \frac{e^{ib(aA+k)s}}{s} \, ds$ will $\to 0$ as $k \to \infty$ provided $\mathrm{Im}(b) > 0$, and will $\to 0$ as $k \to -\infty$ provided $\mathrm{Im}(b) < 0$, noting $s > 0$. So consider now the case $\mathrm{Im}(b) = 0$, i.e $b \in \mathbb{R}$. Write

$-ib(aA+k) = \mathrm{Im}(aA)b + (-\mathrm{Re}(aA) - k)bi$

$= x + iy$, with $x = \mathrm{Im}(aA)b \in \mathbb{R}$ and $y = (-\mathrm{Re}(aA) - k)b \in \mathbb{R}$. Expanding in a Taylor series, note

$$E_1(x+iy) = E_1(iy) + E_1'(iy)x + E_1''(iy)\frac{x^2}{2} + \cdots$$

Writing $E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt$, we have

$$E_1'(z) = -\frac{e^{-z}}{z}, \quad E_1''(z) = \frac{e^{-z}z + e^{-z}}{z^2},$$

$$E_1'''(z) = \frac{(-e^{-z}z + e^{-z})z^2 + 2z(e^{-z}z + e^{-z})}{z^4}$$

$$= \frac{-z^3e^{-z} + 3z^2e^{-z} + 2ze^{-z}}{z^4} \quad \text{and so on,}$$

noting for all $n \in \mathbb{N}$, the greatest power of $E_1^{(n)}(z)$ in the numerator is always 1 less than the denominator.

But this means $\lim_{y\to\pm\infty}|E_1^{(n)}(iy)| = 0$,

e.g. $\lim_{y\to\pm\infty}|E_1''(iy)| = \left|\frac{e^{-iy}iy + e^{-iy}}{(iy)^2}\right|$

$\lim_{y\to\pm\infty} \frac{|y|+1}{|y|^2} = 0$. Furthermore, note

$$E_1(iy) = \int_1^\infty \frac{e^{-iys}}{s}\,ds$$

$$= \left[\frac{1}{s}\frac{1}{(-iy)}e^{-iys}\right]_1^\infty - \int_1^\infty -\frac{1}{s^2}\left(\frac{1}{-iy}\right)e^{-iys}\,ds,$$

so $\lim_{y\to\pm\infty}|E_1(iy)| \leq \lim_{y\to\pm\infty}\left\{\frac{1}{|y|} + \int_1^\infty \frac{1}{|y|}\frac{1}{s^2}\,ds\right\}$

$= \lim_{y\to\pm\infty}\left\{\frac{1}{|y|} + \left[-\frac{1}{s}\right]_1^\infty \cdot \frac{1}{|y|}\right\} = \lim_{y\to\pm\infty}\frac{2}{|y|} = 0.$

So $\lim_{y\to\pm\infty}|E_1(iy)| = 0$. Cool, so this means

$$\lim_{k\to\infty} |E_1(-ib(aA+k))| = \lim_{|y|\to\infty} |E_1(x+iy)|$$
$$\leq \lim_{|y|\to\infty} \left\{ |E_1(iy)| + \sum_{n=1}^{\infty} |E_1^{(n)}(iy)| \left|\frac{x^n}{n!}\right| \right\} = 0.$$

So summarizing, we have the following result/lemma;

$$\lim_{k\to\infty} E_1(-ib(aA+k)) = 0 \quad \text{provided } \mathrm{Im}(b) \leq 0,$$
$$\lim_{k\to-\infty} E_1(-ib(aA+k)) = 0 \quad \text{provided } \mathrm{Im}(b) \geq 0.$$

But this means

$$\lim_{k\to\infty} E_i(ib(aA+k)) = \begin{cases} i\pi & \text{if } \arg(-ib) > \theta_b \\ -i\pi & \text{if } \arg(-ib) < \theta_b \end{cases}$$

provided $\mathrm{Im}(b) \geq 0$, and

$$\lim_{k\to-\infty} E_i(ib(aA+k)) = \begin{cases} i\pi & \text{if } \arg(-ib) < \theta_b, \\ -i\pi & \text{if } \arg(-ib) > \theta_b, \end{cases}$$

provided $\mathrm{Im}(b) \leq 0$.

Now, can we use this result to evaluate the integrals in equation (22)? We will consider each integral carefully one-by-one. To simplify notation, define

$L_1 = \mathcal{X}(1/A)^{1/A} z + x$, $L_2 = -\mathcal{X}(1/A)^{1/A} z + x$. Recall

$$\tilde{\psi} = \frac{-\ \ \delta(\sigma-1)}{B^2 2i}\left\{\int_0^\infty \left(\frac{-1}{(k/A+i)} + \frac{1}{k/A - i}\right)\left(e^{ikL_1} - e^{-z} e^{ikx}\right) dk\right.$$

$$\left. + \int_{-\infty}^0 \left(\frac{-1}{(k/A+i)} + \frac{1}{(k/A-i)}\right)\left(e^{ikL_2} - e^{-z} e^{ikx}\right) dk\right\}.$$

Also recall that for $a, b, A \in \mathbb{C}$,

$A e^{-iabA} Ei(ib(aA+k))$ is an anti-derivative of $\frac{1}{k/A + a} e^{ikb}$. Now, we can express our previous limit result compactly as

$\lim_{k\to\infty} Ei(ib(aA+k)) = \operatorname{sgn}(\arg(-ib) - \theta b) \cdot i\pi$ provided

$\operatorname{Im}(b) \geq 0$, and

$\lim_{k\to-\infty} Ei(ib(aA+k)) = -\operatorname{sgn}(\arg(-ib) - \theta b)\, i\pi$

provided $\operatorname{Im}(b) \leq 0$. Now,

$$\int_0^\infty \frac{-1}{(k/A+i)} e^{ikL_1} dk$$

$$= -A e^{AL_1}\left[Ei(i\underset{L_1 = b}{\underline{L_1}}(\underset{a=i}{\underline{i}}A + k))\right]_0^\infty. \text{ Now,}$$

note that $\operatorname{Im}(L_1) = \operatorname{Im}(\mathcal{X}(1/A)^{1/A} z + x)$

$= \mathcal{X}(1/A) \operatorname{Im}(1/A) \operatorname{sgn}(z) \geq 0$, noting $z \geq 0$.

Note that we need to choose a branch cut to ensure $\lim_{z\to\infty} \int_0^\infty \frac{1}{(h/A + i)} e^{ihL_1}\, dh = 0$.

Note first $\arg(-iL_1) = \operatorname{atan2}(\operatorname{Im}(-iL_1), \operatorname{Re}(-iL_1))$

$= \operatorname{atan2}(-\operatorname{Re}(L_1), \operatorname{Im}(L_1))$

$= \operatorname{atan2}(-\chi(1/A)\operatorname{Re}(1/A)z - x, \chi(1/A)\operatorname{Im}(1/A)z)$.

But $\chi(1/A)\operatorname{Im}(1/A)z > 0 \quad \forall z > 0$, so

$\arg(-iL_1) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \forall z > 0$. Thus if we choose $\theta_b = \frac{\pi}{2}$ we have $\forall z > 0$

$\lim_{h\to\infty} E_i(iL_1(iA + h)) = -i\pi$, noting we already know the solution for $z = 0$. Now

$\lim_{z\to\infty} \int_0^\infty \frac{1}{(h/A + i)} e^{ihL_1}\, dh$

$= \lim_{z\to\infty} -Ae^{(\chi(1/A)z + Ax)} \{ -i\pi$

$- [-E_1(\chi(1/A)z + Ax)$

recall $E_i(z) = -E_1(-z) \pm i\pi$

$- \operatorname{sgn}(\arg(\chi(1/A)z + Ax) - \frac{\pi}{2})\, i\pi] \}$.

Note that $e^{(\chi(1/A)z + Ax)} E_1(\chi(1/A)z + Ax)$

$= \int_1^\infty \frac{e^{(\chi(1/A)z + Ax)(1-s)}}{s}\, ds$.

Now, if $\Re(1/A) > 0$,

$\lim_{z\to\infty} \int_1^\infty \frac{1}{s} e^{(\Re(1/A)z + Ax)(1-s)} = 0$, noting

$1-s<0$. Moreover, $\arg(\Re(1/A)z + Ax)$

$= \operatorname{atan2}(\operatorname{Im}(A)x, \Re(1/A)z + \operatorname{Re}(A)x) \to 0$ as

$z \to \infty$, so $\lim_{z\to\infty} \int_0^\infty \frac{1}{(h/A + i)} e^{ihL_1}\, dh = 0$

as required. Hmm, but note

$\frac{1}{A} = \sqrt{\frac{c^2}{B^2}}$, with $\frac{c^2}{B^2} \in \mathbb{C}$, so we can

simply choose whichever root we like

so that $\Re(1/A) > 0$! So henceforth,

just assume $\Re(1/A) > 0$. Thus

$\int_0^\infty \frac{-1}{(h/A + i)} e^{ihL_1}\, dh = -Ae^{AL_1}\left[-i\pi - E_i^{\pi/2}(-AL_1)\right]$,

where $E_i^{\pi/2}$ denotes E_i with the

branch cut at $\theta_b = \pi/2$. Proceeding

in the same fashion for the remaining

integrals, we find

$\int_0^\infty \frac{+1}{(h/A + i)} e^{-z} e^{ihx}\, dh = Ae^{Ax}\left(-i\pi - E_i^{\pi/2}(-Ax)\right)e^{-z}$,

$$\int_0^\infty \frac{1}{h/A - i} e^{ihL_1} \, dh = A e^{-AL_1} \left(i\pi - E_i^{\pi}(AL_1) \right),$$

$$\int_0^\infty \frac{-1}{h/A - i} e^{-z} e^{ihx} = -A e^{-Ax} \left(i\pi - E_i^{\pi}(Ax) \right) e^{-z}$$

$$\int_{-\infty}^0 \frac{-1}{h/A + i} e^{ihL_2} \, dh = A e^{AL_2} \left(E_i^{\pi}(-AL_2) + i\pi \right)$$

$$\int_{-\infty}^0 \frac{+1}{(h/A + i)} e^{-z} e^{ihx} \, dh = A e^{Ax} \left(E_i^{\pi}(-Ax) + i\pi \right) e^{-z}$$

$$\int_{-\infty}^0 \frac{1}{(h/A - i)} e^{ihL_2} \, dh = A e^{-AL_2} \left(E_i^{\pi/2}(AL_2) + i\pi \right)$$

$$\int_{-\infty}^0 \frac{-1}{(h/A - i)} e^{-z} e^{ihx} = -A e^{-Ax} \left(E_i^{\pi/2}(Ax) + i\pi \right) e^{-z}.$$

Curiously, the sum of these terms can thus be expressed using Chi and Shi integrals, but we will leave in the above form for now! Write $I =$ "the sum of all these terms." Then

$$\Psi = \mathrm{Re} \left\{ \frac{-1}{2\pi B^2 2i} I e^{it} \right\}.$$