Ewan Short 29/8/2025

Non-dimensional governing equations are

$$u_t = \frac{f}{\omega} v - p_x - \frac{\alpha}{\omega} u, \quad (1)$$

$$v_t = \frac{-f}{\omega} u - \frac{\alpha}{\omega} v, \quad (2)$$

$$\frac{\omega^2}{N^2} w_t = b - p_z - \frac{\alpha}{\omega} \frac{\omega^2}{N^2} w, \quad (3)$$

$$b_t + w = Q - \frac{\alpha}{\omega} b, \quad (4)$$

$$u_x + w_z = 0, \quad (5)$$

$$w(z=0) = 0, \quad (6)$$

where ω is a representative frequency, not necessarily the diurnal frequency as in Rotunno (1983) etc. Define ψ such that $(u, w) = (\psi_z, -\psi_x)$. Then (1) and (2) imply

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right) u = \frac{f}{\omega} v - p_x, \quad (7)$$

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right) v = -\frac{f}{\omega} u. \quad (8)$$

Taking $\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)$ of (7) and subbing in (8) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u = -\frac{f^2}{\omega^2}u - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_x. \quad (9)$$

Cross-differentiating (3) and (9) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = \frac{-f^2}{\omega^2}u_z - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_{xz}, \quad (10)$$

$$\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x = b_x - p_{zx}. \quad (11)$$

Using (11) to substitute p_{xz} in (10) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = \frac{-f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)\left(\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x - b_x\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x. \quad (12)$$

Note (4) implies

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b + w = Q$$

$$\Rightarrow \quad -\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x = w_x - Q_x. \quad (13)$$

Subbing (13) into (12) then subbing ψ gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x + w_x - Q_x$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{zz} + \frac{f^2}{\omega^2}\psi_{zz} + \frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{xx} + \psi_{xx} = -Q_x$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\psi_{zz} + \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]\psi_{xx} = -Q_x. \quad (14)$$

Now, for a general forcing Q for which the time dependence is unknown, we can pursue analytic solutions by first taking a Fourier transform from t to σ. Applying this transform to both sides of (14) gives

$$\left[\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\tilde{\Psi}_{zz} + \left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]\tilde{\Psi}_{xx} = -\tilde{Q}_x, \quad (15)$$

where $\tilde{\Psi}$, $\tilde{Q}$ are the Fourier transforms from t to σ of Ψ, Q, respectively. Note because Ψ is real, we have

$$\Psi = \mathrm{Re}\left\{\frac{1}{\pi}\int_0^\infty \tilde{\Psi} e^{i\sigma t}\, d\sigma\right\},$$

so we can assume $\sigma > 0$. Now we can proceed much the same as Rotunno (1983), Qian (2009) etc, taking a Fourier transform of (15) from x to k, to get

$$\left[\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\hat{\Psi}_{zz} - k^2\left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]\hat{\Psi} = -ik\hat{Q}, \quad (16)$$

where $\hat{\Psi}$, $\hat{Q}$ are the transforms of Ψ, Q, respectively. Writing

$$B = \left(-\left(i\sigma + \frac{\alpha}{\omega}\right)^2 - \frac{f^2}{\omega^2}\right)^{\frac{1}{2}}, \quad C = \left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]^{\frac{1}{2}},$$

and $A = \frac{B}{C}$, (16) can be rewritten

$$\hat{\psi}_{zz} + \frac{k^2}{A^2}\hat{\psi} = \frac{ik}{B^2}\hat{Q}. \quad (17)$$

We solve (17) using Green's method. We first solve for the Green's function G satisfying

$$G_{zz} + \frac{k^2}{A^2}G = \delta(z-z'). \quad (18)$$

For $z < z'$ we have

$$G = a_1 e^{imz} + a_2 e^{-imz}, \quad \text{defining } m = \frac{k}{A}.$$

The boundary condition (6) implies

$$a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$$\Rightarrow G = a_1\left(e^{imz} - e^{-imz}\right) = 2ia_1 \sin(mz)$$

$$= a\sin(mz),$$

defining $a = 2ia_1$ for simplicity.

For $z > z'$ we again have

$$G = b_1 e^{imz} + b_2 e^{-imz},$$ but we also

require $G \to 0$ as $z \to \infty$, noting $\alpha > 0$.

Thus if $\text{Im}(m) > 0$, $b_2 = 0$, but if $\text{Im}(m) < 0$, $b_1 = 0$. Thus in either case we can write

$$G = b e^{\text{sgn}(\text{Im}(m)) i m z} \quad \text{for some constant } b.$$

To solve for a and b, note continuity of w at z' implies G must be continuous at z'. Furthermore, (18) implies

$$\lim_{\varepsilon \to 0} \int_{z'-\varepsilon}^{z'+\varepsilon} G_{zz} + \frac{h^2}{\lambda^2} G \, dz = 1$$

$$\Rightarrow \lim_{z \to z'^+} G_z - \lim_{z \to z'^-} G_z = 1.$$

Thus

$$a \sin(m z') = b e^{\text{sgn}(\text{Im}(m)) i m z'}, \quad (19)$$

$$b \,\text{sgn}(\text{Im}(m)) i m e^{\text{sgn}(\text{Im}(m)) i m z'} - am \cos(m z') = 1 \quad (20)$$

Taking $\text{sgn}(\text{Im}(m)) i m \times (19) - (20)$ gives

$$a \,\text{sgn}(\text{Im}(m)) i m \sin(m z') - 1 - am \cos(m z') = 0$$

$$\Rightarrow a\left(im \sin(\text{sgn}(\text{Im}(m)) m z') - am \cos(\text{sgn}(\text{Im}(m)) m z')\right) = 1$$

$$= -ame^{-\text{sgn}(\text{Im}(m)) i m z'} = 1$$

$$\Rightarrow \quad a = -\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} .$$

Subbing a back into (19),

$$-\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} \sin(m z') = b e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'}$$

$$\Rightarrow \quad b = -\frac{1}{m} \sin(m z') . \quad \text{So}$$

$$G = \begin{cases} -\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} \sin(m z) , & z \leq z' , \\ -\frac{1}{m} \sin(m z') e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z} , & z > z' . \end{cases}$$

We then have

$$\hat{\psi} = \int_0^\infty \frac{ik}{B^2} \hat{Q}(z') G(z, z') \, dz' ,$$

$$\tilde{\psi} = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{\psi} e^{ikx} \, dk ,$$

$$\psi = \operatorname{Re}\left\{ \frac{1}{\pi} \int_0^\infty \tilde{\psi} e^{i\sigma t} \, d\sigma \right\} = \operatorname{Re}\left\{ \frac{1}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \hat{\psi} e^{i(kx + \sigma t)} \, dk \, d\sigma \right\} .$$

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<u>Gaussian Point Forcing</u>

Consider $Q = \delta(x)\delta(z-z_f)e^{-t^2}$, $z_f > 0$.

Then $\tilde{Q} = \delta(x)\delta(z-z_f)\sqrt{\pi}\, e^{-\frac{\sigma^2}{4}}$,

$\hat{Q} = \delta(z-z_f)\sqrt{\pi}\, e^{-\frac{\sigma^2}{4}}$, So

$$\hat{\psi} = \begin{cases} -\frac{ik\sqrt{\pi}}{B^2 m} e^{-\frac{\sigma^2}{4}} e^{\operatorname{sgn}(\operatorname{Im}(m))\, i m z_f} \sin(mz), & z \leq z_f, \\ -\frac{ik\sqrt{\pi}}{B^2 m} e^{-\frac{\sigma^2}{4}} \sin(m z_f)\, e^{\operatorname{sgn}(\operatorname{Im}(m))\, i m z}, & z > z_f. \end{cases}$$

For convenience, let $\chi(z) = \operatorname{sgn}(\operatorname{Im}(z))$

for $z \in \mathbb{C}$, and note $\chi(m) = \operatorname{sgn}(k)\chi(\frac{1}{A})$.

Now, while we can of course solve for

ψ more directly for this choice of Q,

let's use our general Green's function

approach detailed earlier. We have for $z < z_f$,

$$\tilde{\psi} = -\frac{iA\sqrt{\pi}}{2\pi B^2} e^{-\frac{\sigma^2}{4}} \int_{-\infty}^{\infty} e^{\operatorname{sgn}(k)\chi(\frac{1}{A})\, i m z_f} \left(\frac{e^{imz} - e^{-imz}}{2i}\right) e^{ikx}\, dk$$

$$= \frac{-A e^{-\frac{\sigma^2}{4}}}{4\sqrt{\pi} B^2} \int_{-\infty}^{\infty} e^{ik\left[\frac{1}{A}\left(\operatorname{sgn}(k)\chi(\frac{1}{A}) z_f + z\right) + x\right]} - e^{ik\left[\frac{1}{A}\left(\operatorname{sgn}(k)\chi(\frac{1}{A}) z_f - z\right) + x\right]}\, dk$$

$$= \frac{-A e^{-\frac{\sigma^2}{4}}}{4\sqrt{\pi} B^2} \left\{ \int_0^{\infty} e^{ik\left[\frac{1}{A}\left(\chi(\frac{1}{A}) z_f + z\right) + x\right]} - e^{ik\left[\frac{1}{A}\left(\chi(\frac{1}{A}) z_f - z\right) + x\right]}\, dk \right.$$

$$+\int_{-\infty}^{0} e^{ih\left[\frac{1}{A}\left(-\chi\left(\frac{1}{A}\right)z_f + z\right) + x\right]} - e^{ih\left[\frac{1}{A}\left(-\chi\left(\frac{1}{A}\right)z_f - z\right) + x\right]} \Bigg\} dh \tag{21}$$

To evaluate these integrals, we need to know what each integrand does as $h \to \pm\infty$, as appropriate. Note $\chi\left(\frac{1}{A}\left(\operatorname{sgn}(h)\chi\left(\frac{1}{A}\right)z_f + z\right)\right)$

$$= \chi\left(\tfrac{1}{A}\right)\operatorname{sgn}\left(\operatorname{sgn}(h)\chi\left(\tfrac{1}{A}\right)z_f + z\right)$$

$$= \begin{cases} 1 \cdot \operatorname{sgn}(\overset{>0}{z_f + z}), & \text{if } \chi\left(\frac{1}{A}\right) = 1,\ \operatorname{sgn}(h) = 1, \\ 1 \cdot \operatorname{sgn}(\overset{<0}{-z_f + z}), & \text{if } \chi\left(\frac{1}{A}\right) = 1,\ \operatorname{sgn}(h) = -1, \\ -1 \cdot \operatorname{sgn}(\overset{<0}{-z_f + z}), & \text{if } \chi\left(\frac{1}{A}\right) = -1,\ \operatorname{sgn}(h) = 1, \\ -1 \cdot \operatorname{sgn}(\overset{>0}{z_f + z}), & \text{if } \chi\left(\frac{1}{A}\right) = -1,\ \operatorname{sgn}(h) = -1 \end{cases}$$

$= \operatorname{sgn}(h)$, recalling $z < z_f$. Similarly

$$\chi\left(\tfrac{1}{A}\left(\operatorname{sgn}(h)\chi\left(\tfrac{1}{A}\right)z_f - z\right)\right) = \chi\left(\tfrac{1}{A}\right)\operatorname{sgn}\left(\operatorname{sgn}(h)\chi\left(\tfrac{1}{A}\right)z_f - z\right)$$

$$= \begin{cases} 1 \cdot \operatorname{sgn}(\overset{>0}{z_f - z}), & \text{if } \chi\left(\frac{1}{A}\right) = 1,\ \operatorname{sgn}(h) = 1, \\ 1 \cdot \operatorname{sgn}(\overset{<0}{-z_f - z}), & \text{if } \chi\left(\frac{1}{A}\right) = 1,\ \operatorname{sgn}(h) = -1, \\ -1 \cdot \operatorname{sgn}(\overset{<0}{-z_f - z}), & \text{if } \chi\left(\frac{1}{A}\right) = -1,\ \operatorname{sgn}(h) = 1, \\ -1 \cdot \operatorname{sgn}(\overset{>0}{z_f - z}), & \text{if } \chi\left(\frac{1}{A}\right) = -1,\ \operatorname{sgn}(h) = -1, \end{cases}$$

$= \operatorname{sgn}(h)$. But this means the first integrand in (21) $\to 0$ as $h \to \infty$,

and the second integrand $\to 0$ as $h \to -\infty$.

So from (21) we obtain

$$\tilde{\Psi} = \frac{-Ae^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\left\{\frac{-1}{i\left[\frac{1}{A}\left(\chi\left(\frac{1}{A}\right)z_f+z\right)+x\right]} - \frac{-1}{i\left[\frac{1}{A}\left(\chi\left(\frac{1}{A}\right)z_f-z\right)+x\right]}\right.$$
$$\left. + \left[\frac{1}{i\left[\frac{1}{A}\left(-\chi\left(\frac{1}{A}\right)z_f+z\right)+x\right]} - \frac{1}{i\left[\frac{1}{A}\left(-\chi\left(\frac{1}{A}\right)z_f-z\right)+x\right]}\right]\right\}.$$

Now, for $z > z_f$,

$$\tilde{\Psi} = \frac{-Ae^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\int_{-\infty}^{\infty}\left(e^{imz_f} - e^{-imz_f}\right)e^{\operatorname{sgn}(h)\chi\left(\frac{1}{A}\right)imz}\,e^{ihx}\,dh$$

$$= \frac{-Ae^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\int_{-\infty}^{\infty} e^{ih\left[\frac{1}{A}\left(z_f+\operatorname{sgn}(h)\chi\left(\frac{1}{A}\right)z\right)+x\right]} - e^{ih\left\{\frac{1}{A}\left[-z_f+\operatorname{sgn}(h)\chi\left(\frac{1}{A}\right)z\right]+x\right\}}\,dh.$$

Note $\chi\left(\frac{1}{A}\left(z_f + \operatorname{sgn}(h)\chi\left(\frac{1}{A}\right)z\right)\right)$

$$= \chi\left(\tfrac{1}{A}\right)\operatorname{sgn}\left(z_f + \operatorname{sgn}(h)\chi\left(\tfrac{1}{A}\right)z\right)$$

$$= \begin{cases} 1\cdot\operatorname{sgn}(\overbrace{z_f+z}^{>0}) & \text{if } \chi(1/A)=1,\ \operatorname{sgn}(h)=1, \\ 1\cdot\operatorname{sgn}(\overbrace{z_f-z}^{<0}) & \text{if } \chi(1/A)=1,\ \operatorname{sgn}(h)=-1, \\ -1\cdot\operatorname{sgn}(\overbrace{z_f-z}^{<0}) & \text{if } \chi(1/A)=-1,\ \operatorname{sgn}(h)=1, \\ -1\cdot\operatorname{sgn}(\overbrace{z_f+z}^{>0}) & \text{if } \chi(1/A)=-1,\ \operatorname{sgn}(h)=-1 \end{cases}$$

$= \operatorname{sgn}(h)$, recalling $z > z_f$.

Also, $\chi(1/A)\,\mathrm{sgn}(-z_f + \mathrm{sgn}(h)\chi(1/A)z)$

$$= \begin{cases} 1\cdot \mathrm{sgn}(\underbrace{-z_f + z}_{>0}), & \text{if } \chi(1/A)=1,\ \mathrm{sgn}(h)=1, \\ 1\cdot \mathrm{sgn}(\underbrace{-z_f - z}_{<0}), & \text{if } \chi(1/A)=1,\ \mathrm{sgn}(h)=-1, \\ -1\cdot \mathrm{sgn}(\underbrace{-z_f - z}_{<0}), & \text{if } \chi(1/A)=-1,\ \mathrm{sgn}(h)=1, \\ -1\cdot \mathrm{sgn}(\underbrace{-z_f + z}_{>0}), & \text{if } \chi(1/A)=-1,\ \mathrm{sgn}(h)=-1 \end{cases}$$

$= \mathrm{sgn}(h)$, recalling $z > z_f$. Thus the respective integrands $\to 0$ as $h \to \pm\infty$, and we find

$$\tilde{\psi} = \frac{-Ae^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\left\{\frac{-1}{i[\frac{1}{A}(z_f + \chi(1/A)z)+x]} - \frac{-1}{i[\frac{1}{A}(-z_f+\chi(1/A)z)+x]} + \frac{1}{i[\frac{1}{A}(z_f - \chi(1/A)z)+x]} - \frac{1}{i[\frac{1}{A}(-z_f-\chi(1/A)z)+x]}\right\}.$$

Note we can then obtain for $z \le z_f$

$$\tilde{u} = \frac{\partial\tilde{\psi}}{\partial z} = \frac{iAe^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\left\{\frac{1/A}{(\frac{1}{A}(\chi(1/A)z_f + z)+x)^2} \frac{1/A}{(\frac{1}{A}(\chi(1/A)z_f - z)+x)^2} + \frac{-1/A}{(\frac{1}{A}(-\chi(1/A)z_f + z)+x)^2} - \frac{1/A}{(\frac{1}{A}(-\chi(1/A)z_f - z)+x)^2}\right\}$$

$$\tilde{w} = -\frac{\partial\tilde{\psi}}{\partial x} = \frac{-iAe^{-\sigma^2/4}}{4\sqrt{\pi}B^2}\left\{\frac{+1}{(\frac{1}{A}(\chi(1/A)z_f+z)+x)^2} - \frac{1}{(\frac{1}{A}(\chi(1/A)z_f-z)+x)^2} - \frac{1}{(\frac{1}{A}(-\chi(1/A)z_f+z)+x)^2} + \frac{1}{[\frac{1}{A}(-\chi(1/A)z_f-z)+x]^2}\right\}$$

Similarly, for $z > z_f$,

$$\tilde{u} = \frac{\partial \tilde{\psi}}{\partial z} = \frac{i A e^{-\sigma^2/4}}{4\sqrt{\pi}\, B^2} \left\{ \frac{\frac{1}{A}\chi(1/A)}{\left(1/A\left(z_f + \chi(1/A) z\right) + x\right)^2} - \frac{1/A\, \chi(1/A)}{\left(1/A\left(-z_f + \chi(1/A) z\right) + x\right)^2} + \frac{1/A\, \chi(1/A)}{\left(1/A\left(z_f - \chi(1/A) z\right) + x\right)^2} - \frac{1/A\, \chi(1/A)}{\left(1/A\left(-z_f - \chi(1/A) z\right) + x\right)^2} \right\}$$

and

$$\tilde{w} = -\frac{\partial \tilde{\psi}}{\partial x} = \frac{-i A e^{-\sigma^2/4}}{4\sqrt{\pi}\, B^2} \left\{ \frac{1}{\left(1/A\left(z_f + \chi(1/A) z\right) + x\right)^2} - \frac{1}{\left(1/A\left(-z_f + \chi(1/A) z\right) + x\right)^2} - \frac{1}{\left(1/A\left(z_f - \chi(1/A) z\right) + x\right)^2} + \frac{1}{\left(1/A\left(-z_f - \chi(1/A) z\right) + x\right)^2} \right\}.$$